Exponential and Logistic Modeling

Exponential Population Model:

If a population P is changing at a constant percentage rate r each year, then

$$\boxed{P(t) = P_0(1+r)^t}$$

where P_0 is the initial population, r is the rate expressed as a decimal, and t is time in years.

Note: If r > 0 *then* $(1+r)$ *is the* growth *factor, if* r < 0 *then* $(1+r)$ *is the* decay *factor for the population.*

EX#1: Tell whether the population model is an exponential growth function or an exponential decay function, and find the constant percentage rate of growth or decay.

A. **Surfside:** $P(t) = 102{,}324(1.025)^t$

$1 + r = 100\% + .025$ or 2.5% growth

B. **Mountain View:** $P(t) = 86{,}527(0.92)^t$

$1 + r = .92 \Rightarrow 100\% - 8\%$ or 8% decay

Exponential Growth and Decay Models

EX #2: Determine the exponential function with the initial value $P_0 = 12$. Increasing at a rate of 6% per year.

$$P(t) = P_0(1+r)^t$$

$$= 12(1+6\%)^t$$

$$P(t) = 12(1.06)^t$$

EX #3: Suppose a culture of 100 bacteria is put into a petri dish and the culture doubles every hour. Predict when the number of bacteria will be 250,000?

Solve algebraically: Use $P(t) = P_0(1+r)^t$

$P(t) = 100(2)^t$

$\frac{250{,}000}{100} = \frac{100(2)^t}{100}$

$2500 = 2^t$

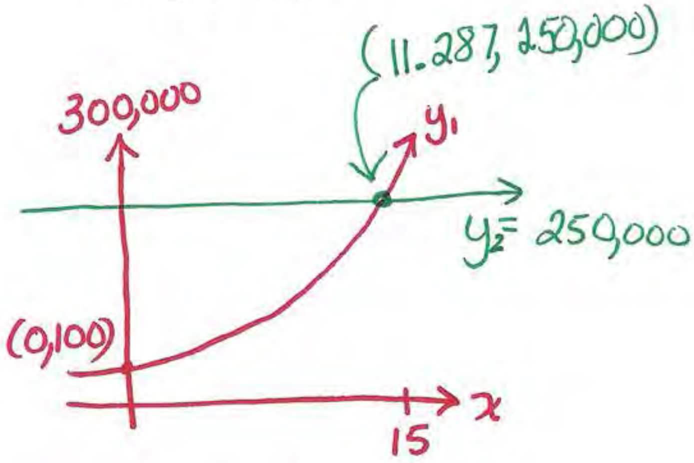
$\ln 2500 = \ln 2^t$

$\ln 2500 = t \cdot \ln 2$

$t = \frac{\ln 2500}{\ln 2}$

$\underline{t \approx 11.287 \text{ hrs.}}$

Solve graphically:



$\left.\begin{array}{l} y_1 = 100(2^x) \\ y_2 = 250{,}000 \end{array}\right\}$ Intersects at (11.287, 250,000)

Modeling Radioactive Decay

Ex #4: Suppose the half-life of a certain radioactive substance is 40 days and there are 8 grams present initially. Find the time when there will be 1 g (gram) of the substance remaining.

Remember: $P(t) = P_0(1+r)^t$

$P(0) = 8$

base $= 1 + r$

decay $= 1 - \frac{1}{2}$
$= \frac{1}{2}$

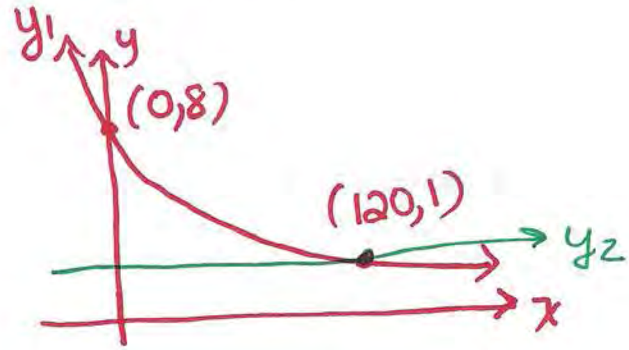
$P(t) = 8\left(\frac{1}{2}\right)^{t/40}$

* $t/40$ = number of half-lives

calculator:

$y_1 = 8(.5)^{x/40}$

$y_2 = 1$



1 gram remains at 120 days

Uninhibited Growth of Cells Formula:

$$A(t) = A_o e^{kt};\ k > 0$$

k is a positive constant that represents the growth rate of the cells.

EX #5: A culture of bacteria follows the law of uninhibited growth. If 500 bacteria are present initially and there are 800 after 1 hour.

A. How many will be present in the culture after 5 hours?

1) $(0, 500)$ $A_o = 500$
$(1, 800)$

2) $A(t) = 500e^{kt}$

$800 = 500e^{1k}$

$\frac{800}{500} = e^k$

$\text{Ln } 8/5 = \text{Ln } e^k$

$\text{Ln}(\frac{8}{5}) = k \cdot \cancel{\text{Ln}}e$

2) $k = \text{Ln } 8/5$

$k \approx 0.4700$

3) $A(t) = 500e^{.47t}$

$A(5) = 500e^{(5)(.47)}$

$A(5) \approx 5242$ bacteria

B. How long before the culture contains 20,000 bacteria?

model : $A(t) = 500e^{.4700t}$

$20{,}00\cancel{0} = 50\cancel{0}e^{.4700t}$

$\text{Ln } e = 1$

$40 = e^{.4700t}$

$\text{Ln } 40 = \text{Ln } e^{.4700t}$

$\text{Ln } 40 = .4700t$

$t = \frac{\text{Ln } 40}{.4700}$

$t \approx 7.848 \text{ hrs}$

$\frac{.848}{100} = \frac{x}{60}$

$x \approx 51 \text{min}$

At about 7hrs 51min there will be 20,000 bacteria

Using Regressions to Model Decay

EX #6: A chemist has a 100-gram sample of a radioactive substance. The table below shows the amount of radioactive material at the end of the week for a 6-week period.

Radioactive Substance	
Week	**Weight (in grams)**
0	100.0
1	90.4
2	76.8
3	69.7
4	58.2
5	49.7
6	44.2

A. Use a graphing utility to fit an exponential function to the data. Write the equation for the data.

$$y = 100.367(0.879)^x$$

B. What is the half-life of the radioactive material?

$$50 = 100.367(0.879)^x$$
$$.49817 = .879^x$$
$$\frac{\ln(.49817)}{\ln(.879)} = x$$
$$\underline{\underline{x \approx 5.4 \text{ weeks}}}$$

C. When will there be 10-grams of material left?

$$10 = 100.367(.879)^x$$
$$x = \frac{\ln(.0996)}{\ln(.879)}$$
$$\underline{\underline{x \approx 17.88 \text{ weeks}}}$$

D. How much radioactive material will be left after 52 weeks?

$$y = 100.367(0.879)^x$$
$$y = 100.367(.879)^{52}$$
$$\underline{\underline{y \approx 0.123 \text{ gram}}}$$

Modeling a Rumor

EX #7: Surfside High School has 1500 students. Max, Patti, Juan, and Andrea start a rumor, which spreads logistically so that $S(t) = \dfrac{1500}{1+38e^{-0.9t}}$ models the number of students who have heard the rumor by the end of t days, where $t = 0$ is the day the rumor begins.

A. How many students have heard the rumor by the end of day 0?

$$S(0) = \frac{1500}{1+38e^{0}}$$

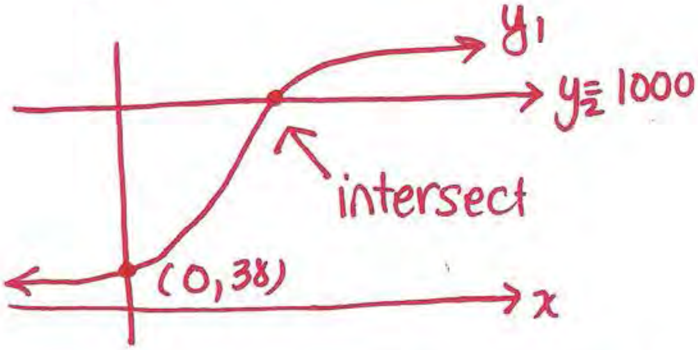
$S(0) \approx 38$ students

$$S(0) = \frac{1500}{39}$$

B. How long does it take for 1000 students to hear the rumor?

$$y_1 = S(x) = \frac{1500}{1+38e^{-.9x}}$$

$$y_2 = 1000$$



(4.81, 1000)
intersection

By the end of the 5th day, 1000 students have heard the rumor.